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# Bose – Einstein Distribution law

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Particles which are Identical, Indistinguishable, Integral or zero spin – (Boson)

Examples: photons, phonons, $\alpha$-particles, They do not obey Pauli's Exclusion Principle.

Consider system of Bosons in thermal equillibrium at temperature 'T'. Let U be the energy of the system and N the number of particles in system. both U and N are constant.

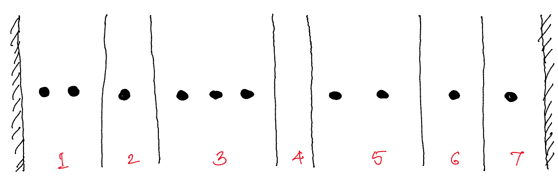
Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be the energies of different particle states in increasing order and let $g_1, g_2, \dots$ be their degeneracies respectively. Let $n_1$ be the particles in energy state $\epsilon_1$, and so on. so the conservation of particles demands that.

$$n_1 + n_2 + n_3 + \cdots = \sum n_i = N = \text{constant.}$$

The conservation of energy says

$$\sum n_i \epsilon_i = U = \text{constant.}$$

Number of ways in which $n_i$ particles may be distributed in energy state $\epsilon_i$ which has degeneracy '$g_i$'.




$n_i = 10$

$g_i = 7$

Note that $(g_i - 1)$ objects are required to split $n_i$ particles into $g_i$ compartments.

Therefore there are $(n_i + g_i - 1)!$ possible permutations that we can make within $n_i + g_i - 1$ objects, but all these permutations are not distinct because $n_i!$ permutations of $n_i$ particles among themselves and $(g_i - 1)!$ permutations of $g_i - 1$ partitions among themselves do not affect distribution and they are irrelevant.

$\therefore$ The number of distinct ways of arranging $n_i$ bosons in $g_i$ quantum state is given by

$$\frac{(n_i + g_i - 1)!}{n_i!\,(g_i - 1)!}$$

It holds for all values of 'i' so the total number of microstates accessible to the system for this distribution is given by.

$$\Omega = \prod_i \frac{(n_i + g_i - 1)!}{n_i! + (g_i - 1)!}$$

$\prod$ stands for product of function over 'i'

$\prod_i$ stands for product of function over 'i'

$\therefore$ Probability of distribution

$$W = \frac{\Omega}{\Omega_{total}}$$

$\Omega_{total}$ is the sum of accessible microstates considering all distributions possible for the system.

Most probable distribution:-

$$\ln \Omega = \sum_i \ln (n_i + g_i - 1)! - \sum_i \ln n_i! - \sum_i \ln (g_i - 1)!$$

Using Stirling's approximation $( \ln x! = x \ln x - x )$

$$\ln \Omega = \sum_i (n_i + g_i - 1) \ln (n_i + g_i - 1) - \sum_i (n_i + g_i - 1) - \sum_i n_i \ln n_i + \sum_i n_i$$
$$- \sum_i (g_i - 1) \ln (g_i - 1) + \sum_i (g_i - 1)$$

$$\delta \ln \Omega = \sum_i \ln (n_i + g_i - 1) \delta n_i + \Sigma (n_i + g_i - 1) \delta \ln (n_i + g_i - 1)$$
$$- \Sigma n_i \delta \ln n_i - \Sigma \ln n_i \delta n_i$$

Since $\delta \ln x = \frac{1}{x} \delta x$, we have.

$$\therefore \delta \ln \Omega = \sum_i \ln (n_i + g_i - 1) \delta n_i + \Sigma \delta n_i - \Sigma \delta n_i - \Sigma \ln n_i \delta n_i$$

Neglecting 1 in comparison to $n_i + g_i$ as $n_i + g_i \gg 1$

$$\delta \ln \Omega = \sum_i \ln \frac{n_i + g_i}{n_i} \delta n_i$$

$\therefore$ for most probabe distribution $\delta \ln \Omega = 0$

$$\therefore \sum_i \ln \frac{n_i + g_i}{n_i} \delta n_i = 0$$

All $\delta n_i$ are not independent.

$\therefore \Sigma \delta n_i = 0$ conservation of particles

$\therefore \Sigma \epsilon_i \delta n_i = 0$ conservation of energy

These equations have to be satisfied simultaneously which can be done by Lagrangian Method of undetermined multipliers

$$\therefore \sum_i \left( \ln \frac{n_i+g_i}{n_i} - \alpha - \beta\epsilon_i \right) \delta n_i = 0$$

$$\therefore \ln \frac{n_i+g_i}{n_i} - \alpha - \beta\epsilon_i = 0 \quad \text{for each } i!$$

$$\therefore \ln \frac{n_i+g_i}{n_i} = \alpha + \beta\epsilon_i$$

$$\therefore \frac{n_i+g_i}{n_i} = e^{\alpha+\beta\epsilon_i}$$

$$\therefore 1 + \frac{g_i}{n_i} = e^{\alpha} \cdot e^{\beta\epsilon_i}$$

$$\therefore \frac{g_i}{n_i} = e^{\alpha} \cdot e^{\beta\epsilon_i} - 1$$

$$\therefore n_i = \frac{g_i}{e^{\alpha} e^{\epsilon_i/kT} - 1} \qquad \left(\because \beta = \frac{1}{kT}\right)$$

$\therefore$ This is BE distribution law.

$$\therefore n(\epsilon)\, d\epsilon = \frac{g(\epsilon)\, d\epsilon}{e^{\alpha} e^{\epsilon/kT} - 1}$$

Where $n(\epsilon)\,d\epsilon$ is the number bosons having energy between $\epsilon$ & $\epsilon + d\epsilon$.